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Le fascicule E du Guide pédagogique en mathématique vient appuyer le programme proposé pour l'enseignement primaire en reprenant, dans un contexte plus particulier et davantage appliqué, les objectifs du programme. Il est destiné à soutenir l'action des enseignants et des autres personnes intéressées à l'apprentisssge des mathématiques. On y retrouve des suggestions, des recommandations, des commentaires, des affirmations, des exemples, des illustrations. Le guide comprend six chapitres: 1. les orientations générales; 2. le concept de fraction; 3. l'égalité et l'inégalité des fractions; 4. les décimales; 5. les opérations faites avec les fractions; 6. les opérations faites avec les décimales.

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Curriculum Guide

ELEMENTARY SCHOOL

MATHEMATICS

BOOKLET E

FRACTIONS

Québec

EDUQ 8870362

Curriculum Guide

ELEMENTARY SCHOOL

MATHEMATICS

BOOKLET E

FRACTIONS

Direction générale du développement pédagogique
Direction de la formation générale

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FOREWORD

This booklet entitled *Fractions* is one of the series of booklets which constitute the elementary school mathematics curriculum guide for teachers and consultants.

The teaching of fractions in the elementary school still gives rise to a number of questions that are not always easy to answer. Though there seems to be some agreement that the set of rational numbers should be treated as an extension of that of natural numbers, opinions vary considerably concerning the manner in which this extension should be presented in elementary schools.

The problem becomes more complex when coupled with considerations related to terminology and the relative importance to be given to ordinary fractions and decimal numbers. In keeping their own orientation, teachers often choose to teach or not to teach certain operations, depending on the algorithms they favour.

The writing of a curriculum guide on such a controversial subject is not an easy task. This work, although not perfect, represents an earnest effort in the right direction. None of the above-mentioned aspects have been neglected and though definite positions have sometimes been adopted, room has also been left for instructional choices.

It is hoped that this curriculum guide will help teachers carry out their task and provide them with a useful tool.

INTRODUCTION

Mere mention of the word "fraction" often generates anxiety in many educators, because it reminds them either of their own difficulties with this concept or of the difficulties in teaching this part of the mathematics program.

Fractions include a set of numbers the learning of which involves a considerable length of time that extends beyond elementary school. This theme is an important part of the program; time must therefore be taken to develop a coherent program, on the one hand, and to decide on appropriate teaching strategies, on the other.

Fractions, like natural numbers, are present in everyday life. Children have their first perceptions of them even before entering school. In fact, fractions enter into everything that can be quantified: SI units of measure, most commercial situations, statistics and probabilities, marks on report cards, and so on.

Many kinds of research are being conducted in Québec and elsewhere concerning the teaching of fractions. This research, together with the introduction of SI, has contributed to the improvement of the mathematical content of programs, with the result that a greater accent has been placed on decimal numbers. It has also led to the development of new teaching approaches and the arrival on the market of new instructional materials.

The majority of schools have already made a basic choice concerning the teaching of fractions. The teaching approach chosen may differ from one milieu to another, but is generally based on a set of educational practices that take the needs of the milieu into consideration.

It is not the intention here either to condemn or commend a particular position, or to appraise all possible positions, but rather to determine a framework within which such instructional choices should be made.

In this booklet, an effort has been made to address these concerns and objectives. For the sake of practicality and clarity, concepts with closely related learning processes have been dealt with in different chapters. However, in actual practice teachers must strive for a comprehensive and dynamic view of the teaching of fractions.

* For teaching purposes, fractions should always be written with a horizontal line: $\frac{3}{4}$.

Because of the typographical restrictions resulting from the size of the printing characters used in this booklet, fractions will often be presented here with an oblique stroke: 3/4. However, teachers are asked to use only horizontal lines in their teaching.

GENERAL ORIENTATIONS

1.1 Terminology and Symbolism

The definition and use of symbols has always given rise to considerable controversy in the teaching of mathematics in general, and in the teaching of fractions in particular. The pedagogical problem, however, is not so much the need for a unique and universal set of symbols, as the need for the use of proper expressions in the classroom.

Confusion is caused, for the most part, by the disparity of expressions used in different books to represent a similar situation, and by the changes in these expressions in recent years. The present booklet does not attempt to define a restrictive terminology, but simply to give sufficient information to ensure uniform understanding of a given text.

There is no one set of symbols used in all textbooks, and it is unlikely that there ever will be. For this reason, teachers and pupils will always be faced with a disparity in the term used to describe a similar situation. In the classroom, it is up to teachers to ensure continuity in the pupils' learning by telling them that the symbols may change according to the author. It is also the teachers' responsibility to explain the symbols that they themselves use.

1.1.1 Fractions

A fraction is a relationship between two quantities. Taken in its general sense, the word "fraction" can designate every possible representation of a rational number.

In the narrow sense of the word, "fraction" is limited to the expression of a relation in the form of an ordered pair [a/b or (a,b), or $b \neq 0$]. Each time that it is used alone in this booklet, it will be with this meaning, whether or not the fraction has a value equal to, smaller than, or larger than one.

While learning, pupils will be led to make transformations in the writing of fractions; for example, they will write $1\frac{1}{2} = 3/2$ or $\frac{1}{2} = 2/4$. Very soon, to communicate more easily, they will feel the need to give specific names to the subsets of fractions.

Fractions with a numerator smaller than the denominator are thus called proper or common fractions; they are the subset of the fractions a/b such that $0 \leqslant a/b < 1$. When fractions are equal to or larger than one ($a/b \geqslant 1$), they are usually designated by one of the following expressions: improper fractions or fractional expressions. All fractions of the type $2\frac{1}{2}$ are usually designated by the expression mixed numbers or fractional numbers.

As the expressions have instructional rather than mathematical implications, and as they will only be used occasionally, there is no advantage to describing one as better than the other. Nevertheless, it is preferable that teachers and pupils in a given milieu use only one expression in the classroom to express the same reality.

1.1.2 Decimal Numbers

All numbers expressed in base 10 can be regarded as decimal numbers. In

current usage, however, this expression designates all fractions expressed in decimal notation; a comma separates the whole part of the number from the fractional part [0,4; 3, 42; 10,01; $0,\overline{3}$]. The line above the 3 in $0,\overline{3}$ means that this number is repeated indefinitely (0,333....).

1.1.3 Rational Numbers

Practical numbers are a class of equivalent fractions which can be written in different ways. For example, the expressions ½; 0,5; 4/8; 0,50; 5/10 all represent the same rational number and each one can be used to designate it. However, usually the simplest expression is used, that is, the one in which the numbers are reduced to the lowest common denominator.

1.2 Fractions and Decimal Numbers

The implementation of SI has caused some confusion with regard to the teaching of fractions. An effort will therefore be made here to establish the importance of fractions in relation to decimal numbers.

There is no doubt that decimal numbers, like the subunits in the Imperial System, constitute the key tool for expressing quantities that are not whole; thus, when comparing an object with a given unit of measure, a child used to say that it measured one foot and eight inches long before understanding the expression 1 2/3 feet. The same applies now to the child who says he measures one metre forty.

Decimal numbers also constitute one of the ways in which numbers can be represented. The child should therefore at some point learn to illustrate a situation which requires the use of rational numbers, either by ordered pairs [a/b or (a,b)], by decimal notation, or by percentages; he/she should even be able to convert from one form of writing to the other.

Children cannot learn about decimal numbers and the concept of rational numbers without a good understanding of fractions. If, for example, the teacher chooses to explain fractions by using the fact that twelve of the twenty-four pupils in a classroom are girls, it would be preferable to use the expressions: "half of the pupils," "one half of the class," "one pupil out of two," or "twelve pupils out of twenty-four" rather than the expression 0,5. The first expression indicates a concrete relationship and makes it possible for pupils to visualize the situation, whereas decimal notation is abstract for pupils who are starting to learn fractions. The abstract nature of decimal numbers would be more evident if the teacher referred to 0,375 of his/her pupils instead of nine twenty-fourths (9/24) or three-eighths (3/8).

In teaching, the use of fractions in the a/b form is thus justified because they make it possible to concretely represent the rational numbers inherent in a situation, and lead to a better comprehension of mathematics, in spite of the fact that they are less important for the expression of measurements.

For the same reasons, a systematic study of equivalences and operations on fractions, with emphasis on decimal numbers, should be made. Fractions in ordered pairs are used even in measurement activities: three-quarters of an hour, one metre and a half, a quarter of a litre, etc. In this document, work will be limited to simple and common fractions, and those which are never encountered in everyday life will be avoided.

The learning of decimal numbers, their use to represent fractional

aspects of reality, and the conversion of fractions of the a/b type into decimal numbers (or vice versa) remain objectives of elementary school teaching, but they will not be undertaken before pupils have mastered the concept of fraction.

1.3 Fractions and Intellectual Development

It seems that the concept of number develops slowly and differs from child to child. As a result of Piaget's research, many authors have tried to put in order the difficulties of learning.

The first level of comprehension[1] of fractions, attained at an early age, is characterized by an awareness of the two terms of the fraction: the division of the whole and the addition (putting together) of the parts which result from the division. The child can easily perform one or the other of these operations separately, but not the two together.

For example, when confronted with a set of eight objects, the pupil can identify one quarter of the objects because only one operation (a division into 4 parts) is being performed. If the set has already been subdivided into 4 groups of 2 objects each, children who have attained this first level of understanding can identify the three-quarters (addition of 3 parts).

Later on, usually at the age of nine or ten, they are able to combine these operations of division and addition, and this constitutes a second level of comprehension. This learning must always be accompanied by concrete manipulations if pupils are to understand the concept of equivalent fractions at this stage. Children are able to find three-quarters of a set of eight objects spontaneously and unassisted.

The third level of understanding, generally characterized by the beginning of formal thinking, makes it possible to compare two fractions taken from two different entities. This level is usually reached between eleven and twelve years of age.

Children are soon no longer content to compare quantities or relationships that are determined by a division, but can evaluate the relationship of each fraction to the whole unit. This peak in understanding the concept of rational numbers, in the ability to conceptualize the invariance of a relationship is rarely attained before the age of thirteen or fourteen. For example, a number that represents the third of a set of marbles could be equivalent to a number that represents the third of a flat surface.

This research has no pretence of being the last word on the subject. It may, however, lead to further reflection on the perceptions that a child of a certain age can have of a number such as ½ or 3/4; it may also force teachers to take these aspects into consideration in the development of learning situations and methods.

1.4 Distribution of the Program Content

As previously mentioned a number of positions have been taken in regard to the teaching of fractions. This becomes even more apparent in the

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1. Hétu, Jean-Claude and Michel Desjardins, L'activité mathématique dans l'enseignement des fractions, (Québec: Presses de l'Université du Québec), pp. 39-69.

distribution of the program content for the different age groups.

The following tables should help in understanding the subsequent chapters. Figure 1 illustrates three possible ways of distributing the objectives and content related to the theme of fractions among the different years of elementary school. In the table, a distinction has been made between two types of instructional activities: the dotted section [...] represents exploration activities and the crossed section [XXX] represents more systematic learning situations or applications of concepts. The different categories of objectives correspond by and large to the five subsequent chapters of this booklet. It is up to readers to analyse these suggestions and to draw up their "own plan" by choosing, for each section, the distribution that best meets their needs.

Figure 2 represents, for each age group, the percentage of the overall time available for mathematics that could be allocated to the teaching of fractions. The section with horizontal lines indicates a minimum percentage; the section with vertical lines indicates a maximum percentage. The interval created conforms to the guidelines given in the program. In a particular milieu, the teaching time can evidently be distributed otherwise by delaying or advancing the teaching of one of the concepts. Nevertheless, by the end of the second cycle, it is important that the child have completed all the learning that the program requires.

The table shows that the study of fractions should, on the average, take from 15 % to 20 % of the time. Other time distributions may be justified, and for this reason, the one suggested here is not compulsory. Its main purpose is to stress the importance of evaluating the time allotted to a particular concept in relation to that allocated for the whole program and for interrelating the different themes.

These tables, and more specifically, the options they represent, conform to the specific provisions and the spirit of the program. They do not reflect all possibilities, but the thought processes required to appropriately balance the different learning situations presented to the children. The best distribution will, however, always be the one developed by each "milieu" to meet the specific needs of its pupils.

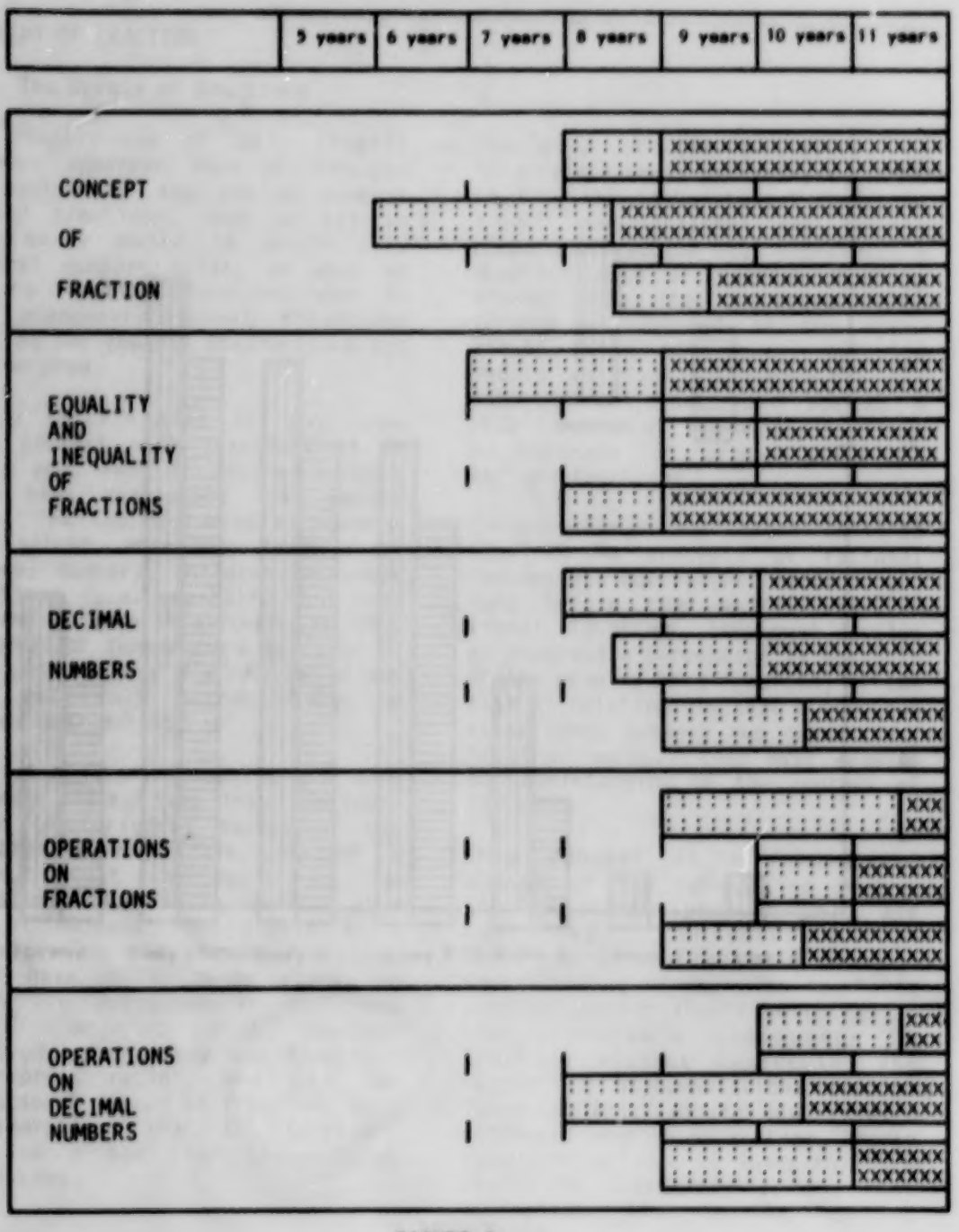

5 years
6 years
7 years
8 years
9 years
10 years
11 years
CONCEPT OF FRACTION
EQUALITY AND INEQUALITY OF FRACTIONS
DECIMAL NUMBERS
OPERATIONS ON FRACTIONS
OPERATIONS ON DECIMAL NUMBERS

FIGURE 1

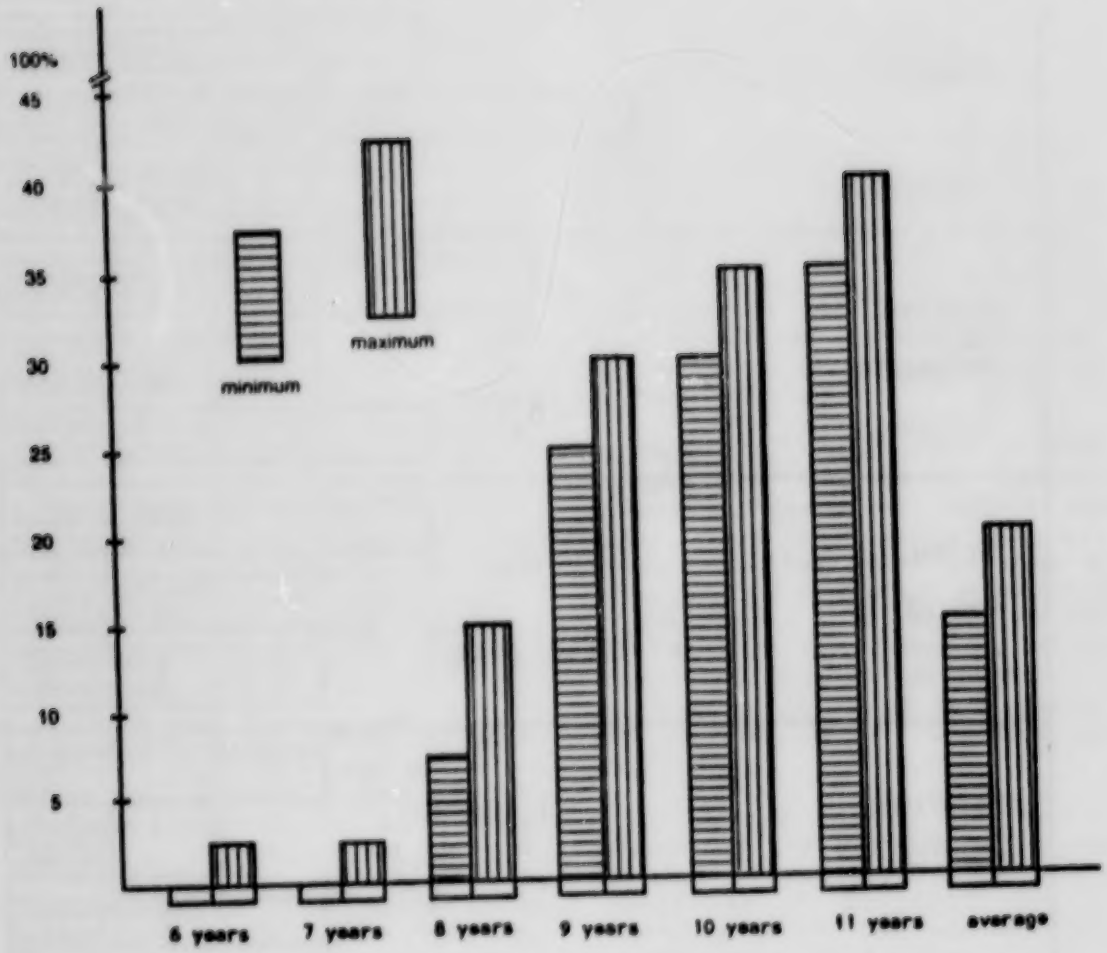

100%
45
40
35
30
25
20
15
10
5
minimum
maximum
6 years
7 years
8 years
9 years
10 years
11 years
average

FIGURE 2

CONCEPT OF FRACTION

2.1 The Origin of Fractions

The importance of this chapter becomes apparent when we consider the origin of the set of numbers called fractions, when we try to imagine a world in which only natural numbers exist, or when we try to identify where and when in the elementary school situations calling for the use of fractions are encountered.

It is certain that children come into contact with fractions at an early age, even if natural numbers have been understood long before then. At the beginning of elementary school, when they learn about natural numbers, children encounter fractions (e.g. one-half) that they regard as quantities much as they conceive of the numbers 1, 2, or 3. Such is the case for the child who gets only half a cookie when he wanted two cookies!

The perceptions of adults are more refined. They take into consideration the original aspect of the fraction, that it is part of a whole, be it the result of the division of a whole (a cake) or of a set of objects (the pupils in a class). They can compare fractions which have unlike denominators, or which are expressed in different ways: ordered pairs, decimal notation, etc. They use fractions to express ratios, and take the operational aspect of fractions into consideration, that is, fractions used as a tool for transforming quantities.

The role of the school is to interrelate these perceptions. This is also the instructional objective sought when children are led to compare a fraction that represents a quantity with a fraction that is broken into its component parts through a sequence of two operations. In this case, priority should be given to approaches and interventions that make use of a wide variety of learning situations or manners of presentation to explain fractions.

Consequently, long before children acquire the concepts of rational numbers and equivalent fractions and long before they master computational algorithms, they must develop as accurate a perception as possible of the meaning of a fraction, of the kind of relationship that a fraction establishes between two quantities, in other words: they must develop an understanding of the concept of fraction.

This concept is a fundamental element of the mathematics program as far as rational numbers are concerned. Children should therefore attain the best possible understanding of it. The teacher's ongoing concern should be to refine the children's conception of fraction without neglecting the acquisition of computational skills. These skills, which may require different degrees of mastery, are an important adjunct to learning which should be introduced at the most appropriate time in the learning process.

2.2 Fractions as Numbers

Faced with situations that involve simple fractions, children can usually express these fractions as a ratio: a half, a third, one-half, one-quarter, one object out of four, etc.

FIGURE 3

The remark "half of the children are girls!" with regard to the situation in this picture or to a similar one, illustrates a descriptive process. The fraction used in this case is static in nature. It represents a quantity or a FRACTIONAL STATE.

A fraction can evidently represent a quantity that is larger or smaller than one; however, it is normally used to designate quantities smaller than one. In other words, the fraction usually designates part of an object or of a set of similar objects, as demonstrated in the following examples.

FIGURE 4

If an apple is cut into two parts we obtain two halves, or two half apples: we write $\frac{1}{2}$ an apple, that is, 1 piece out of 2. If we add $\frac{1}{2}$ an apple to another $\frac{1}{2}$ an apple, we obtain 1 whole apple; thus, two apples would give us four half-apples.

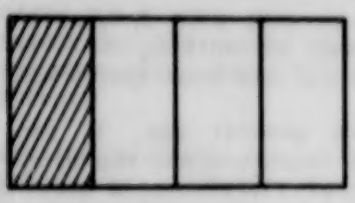


FIGURE 5

"$\frac{1}{4}$ of this figure is hatched." The 4 in $\frac{1}{4}$ means that the figure was separated into 4 equal parts, and the 1 means that only 1 of these parts is hatched.

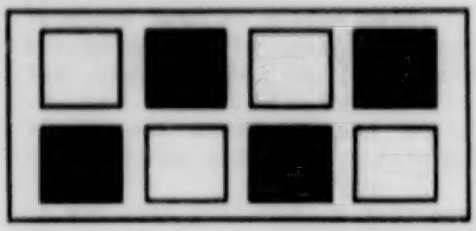


FIGURE 6

"4 of the 8 squares are black." This example enables us to present the roles of the denominator and of the numerator in a different way.

When the teacher wants to help pupils understand the meaning of the fraction as part of a whole, using these or other examples as a point of departure, it might be worthwhile to call their attention to the denominator by first having them divide the unit into equal parts. The numerator then becomes the indicator of a second operation which consists of grouping a certain number of parts.

The two last examples illustrate the distinction between dividing different objects such as fields, cakes and circles into portions (Figure 5), and dividing a set of similar objects into groups (Figure 6). Even though the concept of fraction seems to correspond more often to that illustrated in the first example, there is no indication that one presentation is more advantageous than the other; the two presentations call for the same logical operations. In fact, they are two complementary applications of the concept of fraction and the children should be presented with both types at each stage in the learning of fractions.

This approach to fractions as the division of a unit into portions is an almost inexhaustible source of instructional situations which make it possible to use the child's natural environment, and to foster the skills essential to the learning of this concept. This approach is not new, but it contributes greatly to developing the child's perception both of fractions as a relationship between two numbers, and of the respective roles of the denominator and numerator. For these reasons, it is highly recommended.

During the learning activities, the child tends to make two types of errors in dividing up a unit; either the parts are not equal, or, some elements of the set are neglected. This happens frequently in dividing up a set of similar objects. These errors, which involve the very basis of the concept of fractions, are often caused by distractions and sometimes it is sufficient to direct the child's attention to his work for him/her to rectify his/her errors.

2.3 Fractions as Quotients

The children should be made aware of another meaning of fraction, a meaning that is often used but rarely explained in depth. Fractions can represent the result of a division that is a quotient.

Suppose that 8 chocolate bars are to be divided among 4 people. To determine each person's share, 8 is logically divided by 4. This is easy enough because the operation has 0 as a remainder. On the other hand, if there were only 5 bars the same division would have been performed and 5/4 of a bar obtained for each person; this 5/4, which is another way of writing 5 ÷ 4, is the result of the same operation.

In a more general way, it can be seen that fractions can represent an operation of division or a quotient, the numerator being analogous to the dividend and the denominator to the divisor. When we perform divisions which have remainders, for example 15 ÷ 7, "2 with a remainder of 1" is a good answer but 15/7 is an equally exact and accurate answer, in terms of the actual numbers.

2.4 Fractions as Operators

Besides being numbers that represent whole quantities or parts of quantities, teachers often present fractions as operators that can transform quantities.

These "teacher fractions" operate on the same principles as "function machines"; in ordinary language, they correspond to a command, an order, or an operation that makes it possible to pass from an initial quantity to another quantity.

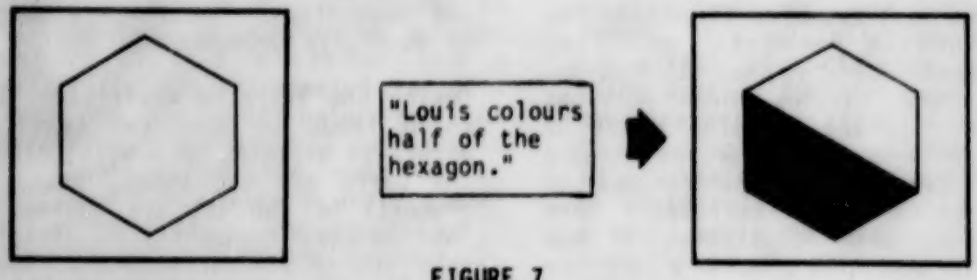


FIGURE 7

For this reason, in many textbooks, the expression "fractional operator" is used to designate this meaning of fraction, in spite of the fact that mathematicians do not accept that a rational number be defined as an operator. It must not be forgotten that this meaning of fraction in essence represents an instructional approach whose objective is to help children develop a better understanding of the concept of fraction. Rational numbers, of which fractions are one representation, are numbers

which have all the concomitant properties; the notion of fractional operator is, then, a teaching concept.

Function machines can utilize whole numbers as well as fractions; in fact, even if the graphic presentations sometimes differ, the principle remains the same.

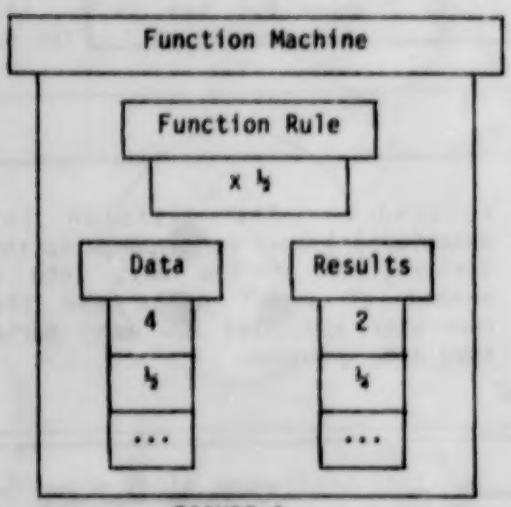


FIGURE 8

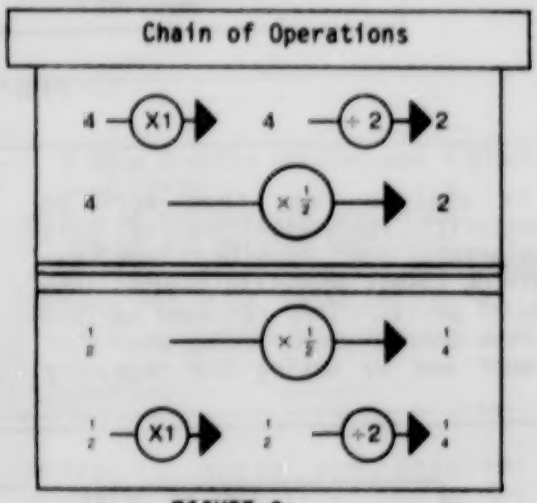


FIGURE 9

In each case, the function rule or operator is x ½. It can be expressed as a succession of two operations, a division and a multiplication represented respectively by the denominator and the numerator. Obviously it is possible to have a function rule or an "operator" involving a whole number or a fraction. The advantage of this teaching/learning process is that it brings all calculations of fractions back to a single operation on whole numbers.

Similarly, it is important to lead children to systematically break fractions down into two operations. This breaking down of fractions greatly simplifies the problem of understanding the concept of fractions; it permits a better understanding of numerator and denominator as analogous to multiplication and division. Indeed, if a multiplication reverses the effect of a division, the same thing applies for the numerator and denominator; the first is analagous to a multiplication and the second, to a division. The attention given to these relationships will provide a solid basis for understanding the operations of multiplication and division.

These operations should first be carried out on concrete materials since it is through hands-on experiences that the children will enrich their comprehension by interpreting the denominator as the division of a unit into equal parts, and the numerator as the number of parts to be grouped.

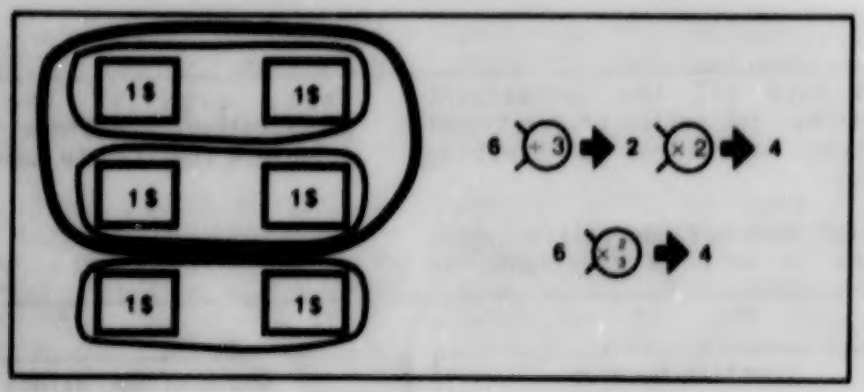


Figure 10

The children can then work on pictorial representations of this material; for example, they can divide common geometric figures into equal parts. They can even perform these operations by using sheets of paper and by making the necessary foldings. The division (or denominator) then corresponds to the dividing up of the unit into a number of equal parts and the numerator indicates how many parts should be grouped.

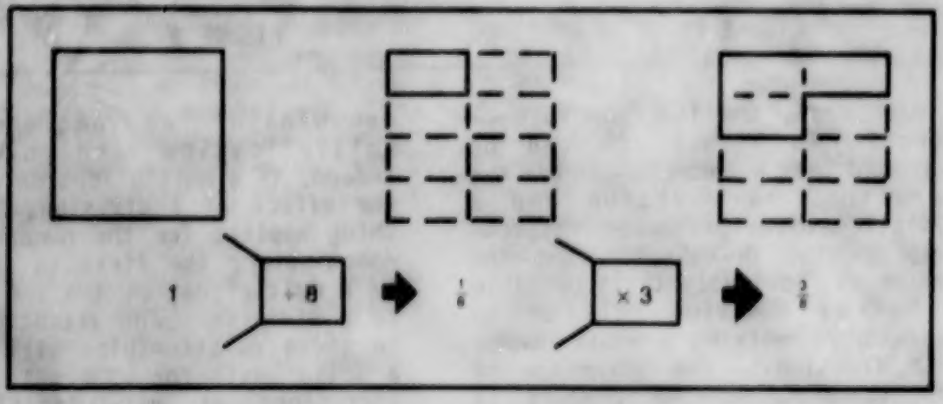


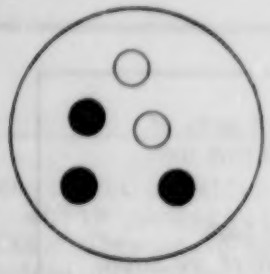
FIGURE 11

It is only after having thoroughly understood these steps that the children will work exclusively on numbers or on "chains of operators." They will be led to deduce the concept of fraction from their hands-on experiences. The last examples given also highlight the importance of the order of operations. When the children perform concrete or semi-concrete operations, when they try to understand the concept of fraction, needless to say the order of the

operations of division and multiplication can be changed at will even if, mathematically, the answer is always the same.

2.5 Development of the Concept of Ratio

The study of the concept of ratio is another means of increasing pupils' understanding of fractions because a ratio can be expressed in different ways, including fractions: 75 %, (3:4), (3,4) and 3/4 express the same ratio. On the other hand, the teacher should be aware that the manner in which certain cases involving ratio are introduced can make them more difficult to understand and even confuse the child as in the following instance:



- I have 2 white marbles and 3 black ones (2/3)
- 2/5 of the marbles are white
- 3/5 of the marbles are black
- I have white marbles and black marbles in a ratio of 2 to 3 (2:3)

How can a child understand that in one situation we compare a group of elements to the set and represent the ratio with a fraction (here: 2/5), and that in another situation we compare a subset to another subset and represent this new ratio by another fraction (in this case, 2 out of 3 or 2/3)?

Care must be taken not to switch from one interpretation to the other indiscriminately and not to introduce the two simultaneously. Consequently, priority should clearly be given to fractions representing part of a whole: 3/5 of the marbles are black.

To ignore the other aspect completely would be no better. This distinction can be introduced and the concept of fraction elucidated by using situations that arise in everyday life. Frequent exposure to this type of situation enables the child to solve the problems presented by such situations by the end of his/her elementary school studies. Furthermore, he/she usually likes to do so. Care should be taken not to choose cases involving the most complex ratios, cases that would surpass the children's comprehension. Making and reading graphs are good ways of visualizing a problem or of examining it from a different angle.

The following cases involve ratios that can be understood by ten- to twelve-year-old pupils:

- I have a bag of marbles that I am dividing up between John and Mary in a ratio of 1 to 3.
- 3/4 of my marbles are red; I have 24, so how many are not red?
- If 4 pencils cost $ 0.30, how much will 10 pencils cost?

- I drove 2½ hours at 80 km. My car uses 6 litres of gas for every 100 km. How many litres did I use?

- The text contained 60 questions and Louis succeeded in 75 % of the questions. How many correct answers did he have?

2.6 Use of the Fraction Chart

Keeping in mind the idea of using the children's environment to help them discover the concept of fraction, fractions could also be treated as parts of numbers.

Cuisenaire rods or other similar materials can be used to make the concept of fraction concrete. By taking one rod and comparing it to another one that is considered to be a unit, a fraction can be determined.

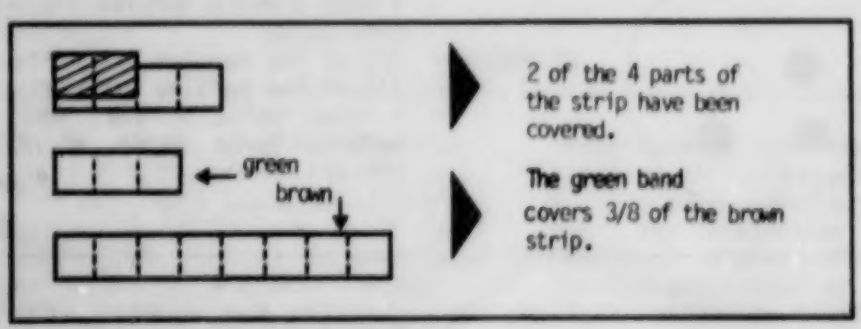


FIGURE 12

The same objective can be attained using the fraction chart because the principle on which it is based is virtually the same as that used with the rods. The children also apply this process when making their own materials or even when working with illustrations; however, the materials will have to be manipulated with great care, if the results of the operations are to be valid.

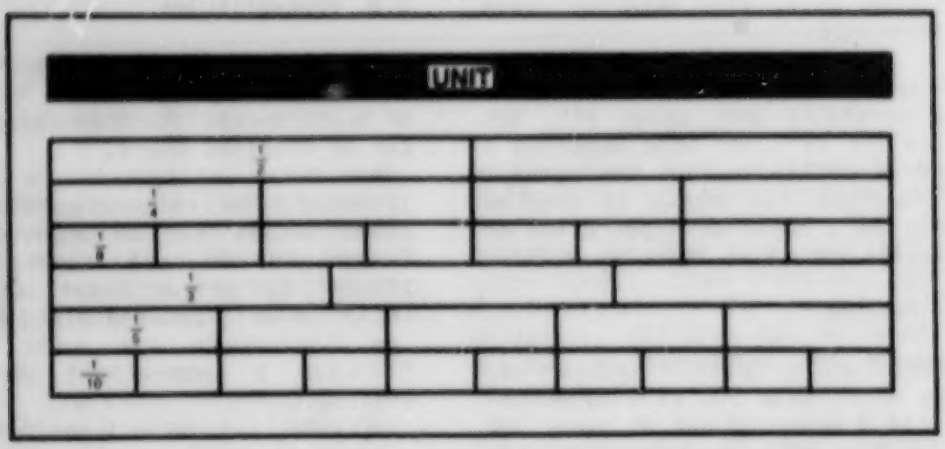


FIGURE 13

Fraction charts usually stop at 1 unit, which presents certain drawbacks. They could equally well be extended to include 2, 3 or 4 units. This drawback is minimized when the charts are used for one of the following purposes: to develop understanding of the concept of fraction (¼ corresponds to 1 out of 4 parts of a preselected unit); to make the respective values of 2 fractions concrete (1/4 1/3); to develop understanding of the meaning of operations on fractions.

2.7 Use of the Number Line

Fraction charts make use of strips of paper to which the value of a unit is arbitrarily assigned. If several units are added one beside the other, something very similar to a number line would be obtained.

The quantities used to teach fractions are usually between zero and 1. Teachers should also bring up situations which involve the use of fractions larger than one in order to help the children understand the infinite nature of this set of numbers. The number line is a good way of ensuring that pupils understand this. The following considerations should, however, be borne in mind to ensure that the point is handled judiciously.

What does a number line mean to children on the conceptual level? It can be used quite effectively in relation to natural numbers: children are able to grasp the one-to-one correspondence between this set of numbers and the set of points on the line. Can they grasp the same relationships between the number line and the set of rational numbers?

Sometimes 1 is said to be the distance between zero and one; ½ therefore represents half this distance. This transition is relatively easy to make. On the other hand, 1 is also said to be the point situated at the end of a unitary distance. In this case, what does ½ represent in the minds of nine- or ten- year-olds? At this point, the different operations

performed using this material could be mentioned.

The number line must therefore be used carefully and, above all, not before the children have acquired a sound understanding of the concept of fraction. It should be treated mainly as a topic of exploration or to foster intuitive learning.

2.8 Percent

Percents are a special category of ratio which have 100 as a denominator; 10 % means 10 out of every 100 or 10 for each hundred (10:100). All problems of percent could be treated as a search for equivalent fractions. For example, to find 8 % of 15, we could say: if I have 8 out of a hundred, how many do I have out of 15?

$$\frac{8}{100} = \frac{x}{15} \quad \text{or} \quad 8 : 100 = x : 15$$

Nevertheless, 10 % is also a notation for 10/100, 1/10 or 0,1. Most business activities involve the use of percents that can be changed into simple fractions (20 %, 10 %, 33 1/3 %, 50 %). Finding the percentage equivalent for a given number is usually considered an operation involving fractions. Percents are a type of ratio that are usually treated as fractions.

$$\frac{8}{100} \times 15 = \frac{8 \times 15}{100} = 1,20$$

Business activities not only illustrate clear applications of ideas related to fractions but also foster the acquisition of a whole vocabulary that the children are somewhat familiar with because of its frequent occurrence: discount, sale, reduction, tax, interest, etc.

2.9 Probabilities

Probabilities is one of the subject areas covered in detail in Booklet G on measurement. Further information can be obtained there.

It should be reiterated that all probabilities can be expressed as fractions in a/b form. The probability of an event occurring is, after all, between 0 and 1.

"I have 1 chance out of 8 of winning..."
((1:8), 1/8, 12.5 %, or 0,125)

"The operation succeeds 9 times out of 10..."

$$\left(\frac{9}{10}\right)$$

Nevertheless, even though a fraction is a good representation of a probability, and even though it is possible to find one probability that is equivalent to another in the same way that an equivalent fraction can be found, it is important to recognize that a fraction representing a probability does not necessarily follow the same rules in terms of operations.

3

EQUALITY AND INEQUALITY OF FRACTIONS

3.1 Equivalent Fractions

In terms of mathematics, a rational number indicates the ratio between two whole numbers. This ratio can be expressed in different ways. Since all fractions express ratios and many ratios can be equal even though they are expressed as different fractions, a set of different fractions expressing the same rational number forms an equivalence class. These fractions are said to be equivalent.

To see rational numbers as an equivalence class demands a capacity for abstract thought which surpasses that of the average child. Thus it is important to develop the concept of fraction (see Chapter 2) without referring to the abstract concept of rational number, which is not, strictly speaking, an objective of mathematics teaching at the elementary school.

On the other hand, children have no difficulty understanding that fractions can represent the same quantity or the same reality, in other words: that fractions can be equivalent. To reach this objective, children can be asked to observe a phenomenon and discover its properties.

The equivalence of two fractions must be something that the pupils build, manipulate, or observe in relation to a preselected unit. As was the case for developing an understanding of the concept of fraction, pupils can discover the concept of equivalent fractions using concrete materials. For example, by folding and cutting a sheet of paper, pupils can discover the equality of $\frac{1}{2}$, and $\frac{2}{4}$ or $\frac{4}{8}$.

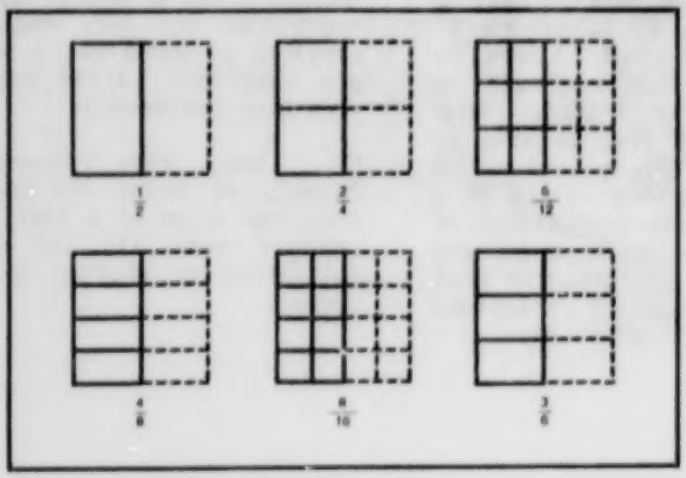


FIGURE 14

The fraction chart presented in the preceding chapter is a very useful instrument for learning about equivalence among different fractions. Children who have already made their own charts can refer to them to compare two or more fractions; if quick glances are not enough, they can measure each fraction with a strip of paper or any other unit of measure.

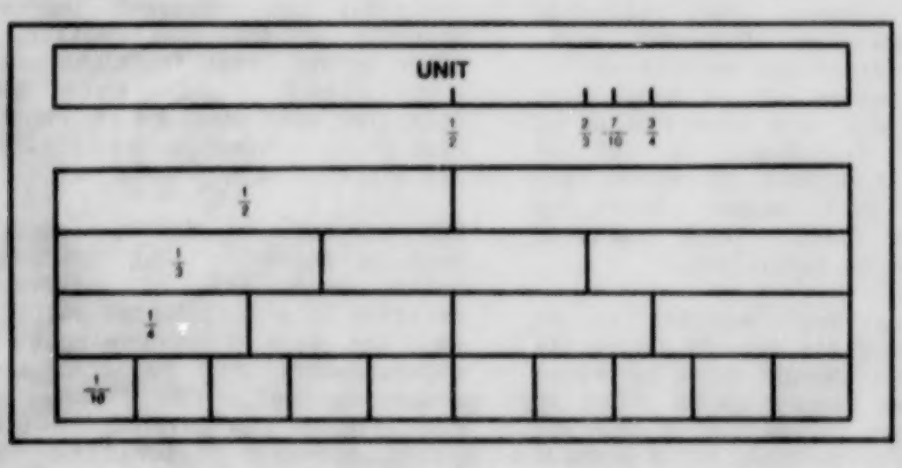


FIGURE 15

This kind of exercise is very important because it enables pupils to move on from manipulating concrete materials to using one of the first forms of symbolic representation. Pupils should be able to identify a fraction from pictures, but must also be able to perform divisions by themselves on their own illustrations, based on a given fraction. The equivalence of fractions thus becomes apparent and the children come to recognize that the list of equivalent fractions could be extended indefinitely.

The multiplication or division of a fraction by an element that is neutral (i.e. a form of 1) for these operations (1, 2/2, 3/3) makes it possible to construct a set of what are sometimes called extension or reduction procedures.

Even though this process is more formal, it helps children realize that the value of a fraction is not changed when its two terms are multiplied or divided by the same number.

$$\frac{1}{2} = \frac{1\times 2}{2\times 2} = \frac{1\times 3}{2\times 3} = \frac{1\times 4}{2\times 4}$$

$$\frac{1}{2} = \frac{2}{4} = \frac{3}{6} = \frac{4}{8}$$

FIGURE 16

Teachers must also remember that an understanding of equivalent fractions (the basis for verifying the equivalence of two fractions) and skill in finding fractions that are equivalent to a given fraction are essential to understanding and learning the addition and subtraction algorithms for fractions. As much time as is needed should be spent on exercises involving the conversion of two fractions to a like denominator.

Converting fractions to decimal notation is also an excellent way of verifying whether two fractions are equal: thus 4/6 and 6/9 are equivalent because both of them can be written as 0,6.

In reference to Section 2.5, two fractions regarded as operators are equivalent if they have the same effect, in other words, if they represent "equivalent functions."

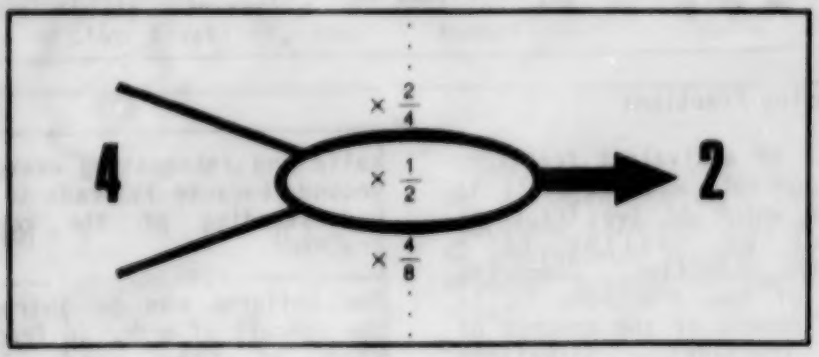


FIGURE 17

Thus, children discover that there are large families of fractions which have the same value and that all the fractions in this set are equivalent. The fraction which is simplest in its expression is usually used to represent this set, in other words, the one whose numerator and denominator have no common factor other than 1.

Nevertheless, it must be noted that each fraction in a set of equivalent fractions can be used to designate this set; thus we can say "The equivalence class 4/8."

Once pupils have understood how two fractions can represent the same quantity, this phenomenon can be illustrated in different ways. The number line is a useful tool in this respect because it exposes children to fractions larger than one. Its use as a means of representing equivalent fractions introduces learners to the concept of rational numbers, even though it is not specifically designated as such. Use of the number line also makes it possible to establish a relationship with measurement.

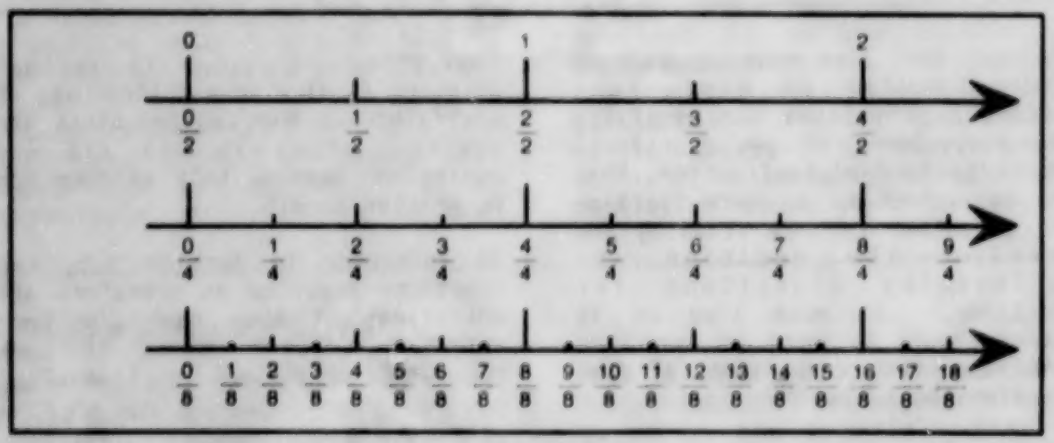


FIGURE 18

3.2 Comparing Fractions

The concept of equivalent fractions is a key concept, whereas skill in determining which of two fractions is larger or smaller is a complementary objective. Comparing the value of two fractions is in fact a refinement of the concept of equivalent fractions. Situations calling for the application of this skill are relatively scarce. An understanding of equivalent fractions does, however, lay the groundwork for other learning (e.g. operations of addition and subtraction). This part of the program should therefore not be neglected: first, because it is a valid and interesting exercise, and second, bacause it leads to a better understanding of the concept of fraction.

The children can be introduced to the concept of order in fractions as soon as they start learning equivalent fractions using the fraction chart. At the moment they discover that two fractions are not equivalent, they automatically observe that one is larger or smaller than the other. It is not easy to compare fractions that have thirds, quarters, or tenths as a denominator. On the other hand,

with the help of Figure 15 on page 18, the following inequalities become immediately apparent.

$$\frac{8}{9} < \frac{7}{10}$$

$$\frac{7}{10} < \frac{3}{4}$$

$$\frac{3}{4} > \frac{2}{3}$$

The number line provides another means of helping children develop a concrete understanding of the respective value of two fractions. For example, the largest fraction of a positive value is the one that is the furthest from zero. However, since it is not always easy to divide a segment of a straight line into equal parts, the number line should not be overused. It is preferable to construct several number lines which are divided into the same type of unit, rather than to overload the one number line with more than one division.

The first comparisons that are made between fractions should involve fractions that have a like denominator. The children will quickly observe that the larger the numerator, the larger the fraction. From then on, they will instinctively try to convert fractions to a common denominator and will use the extension and reduction procedures referred to in the section on equivalent fractions. Converting two fractions to a common denominator is one of the main applications of the concept of the lowest common multiple, which is explained in the booklet on natural numbers.

3.3 Complementary Activities

When the children have thoroughly grasped the concept of equality and inequality between fractions, they are in a position to develop computational skills. To verify the equivalence of two fractions, they will be taught to use the so-called "cross product" technique, which is based on the principle that an equation remains true if all its terms are multiplied by the same number.

2×10	=	4×5
$(2 \times 10) \times 3$	=	$(4 \times 5) \times 3$
THE SAME AS		
$\frac{3}{4}$	=	$\frac{6}{8}$
$\frac{3}{4} \times (4 \times 8)$	=	$\frac{6}{8} \times (4 \times 8)$
3×8	=	4×6

FIGURE 19

On a more formal level, two fractions (3/4 and 6/8) are said to be equivalent if and only if the diagonal product of their terms is equal.

$$\frac{3}{4} \times \frac{6}{8}$$

$$3 \times 8 = 6 \times 4$$

The five charts on pages 23-24 make it possible to identify equivalent fractions. These charts represent ways of arranging numbers so that the equivalent fractions stand out, but they are not all designed to help pupils develop a better understanding of the concept of equivalent fractions.

Figures 20 and 23 present charts that are relatively similar. The equivalent fractions are more evident in the first chart, however, because they are placed on the same line. The advantage of Figure 23 is that potentially it could contain all fractions. In fact, the dots in Figure 20 correspond to the top right-hand corner of each corresponding fraction in Figure 23.

Figure 21 is a fraction chart similar to the one already mentioned. It could be extended infinitely, but it is not necessary to represent all the numbers found in denominators. Usually it is limited to the first multiples of 2, 3 and 5. The children can work on it using strips of paper, strings, or rulers since two fractions are equivalent if they occupy the same space relative to the origin.

In Figure 22, the characteristic of equivalent fractions is that the diagonal lines drawn through their numerators and denominators marked in the horizontal band meet at the same point on the vertical line; the nearer a fraction is to 1, the farther up or down it is on the vertical line. In fact, each fraction equivalent to 1 (1/1, 2/2, etc.) determines a parallel to the vertical line; this is the sense of the word "limit" that appears on the figure.

Figure 24 is a chart of multiples of 2 towards the right and of 3 towards the bottom. The pupils make up their own keys similar to those presented at the bottom of the figure. Regardless of where they are placed, keys make it possible to read equivalent fractions.

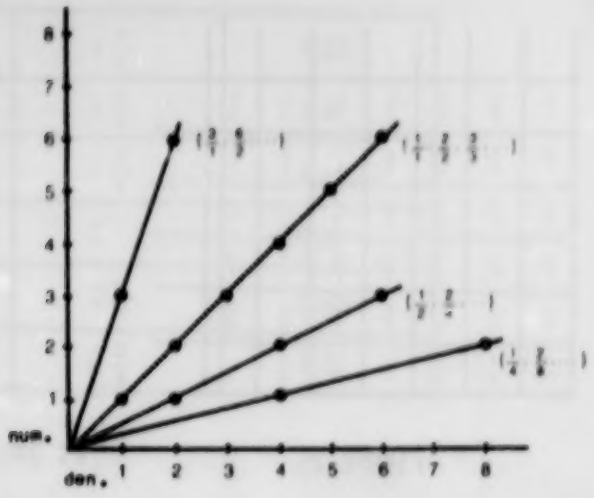


FIGURE 20

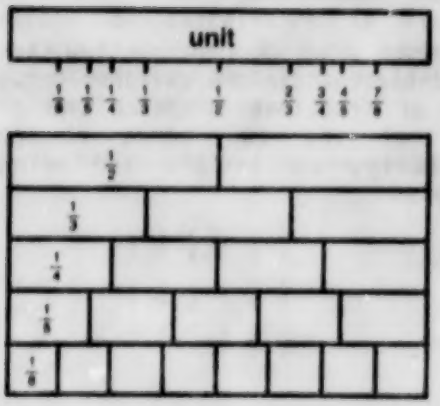


FIGURE 21

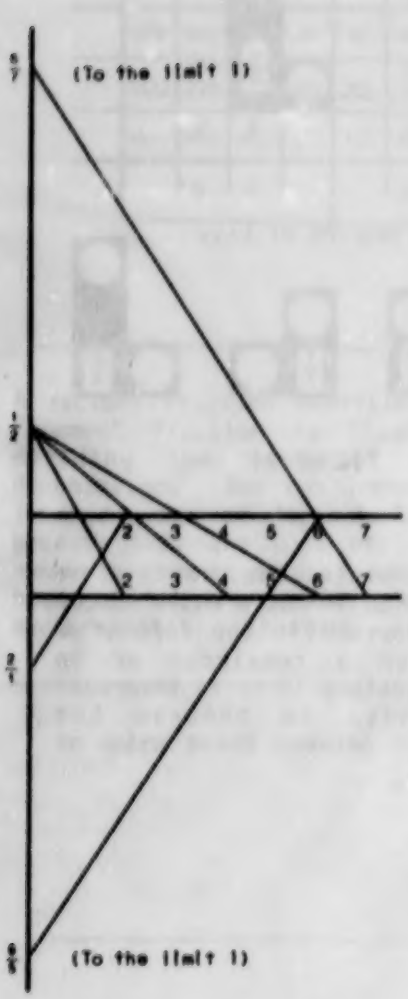


FIGURE 22

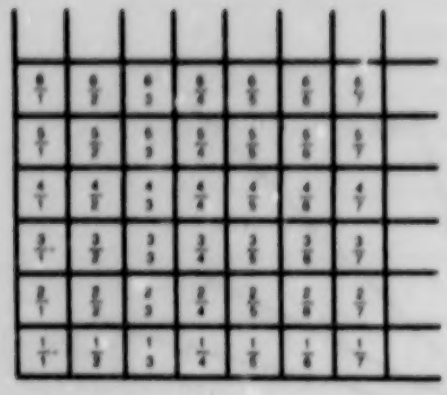


FIGURE 23

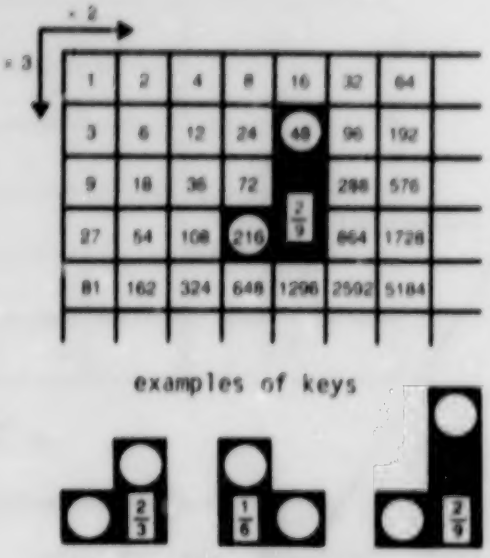


FIGURE 24

It could also be interesting and enriching to have the children order sets of fractions in which one or both of the two terms change regularly, as in the following examples:

$$-\frac{1}{3}\cdot\frac{2}{3}\cdot\frac{3}{3}\cdot\frac{4}{3}\cdot\frac{5}{3}\cdot\frac{6}{3}\cdots$$

$$-\frac{1}{2}\cdot\frac{1}{3}\cdot\frac{1}{4}\cdot\frac{1}{5}\cdot\frac{1}{6}\cdot\frac{1}{7}\cdots$$

$$-\frac{1}{2}\cdot\frac{2}{3}\cdot\frac{3}{4}\cdot\frac{4}{5}\cdot\frac{5}{6}\cdot\frac{6}{7}\cdots$$

Moreover, since the part of a quotient which is not a whole number can be expressed in the form of a fraction or a remainder or in decimal notation, it is an interesting activity to observe the relationship between these modes of expression.

5 ÷ 4	1 R 1	$1\frac{1}{4}$	1,25
6 ÷ 4	1 R 2	$1\frac{2}{4}$	1,5
7 ÷ 4	1 R 3	$1\frac{3}{4}$	1,75
8 ÷ 4	2 R 0	2	2,0
9 ÷ 4	2 R 1	$2\frac{1}{4}$	2,25
10 ÷ 4	2 R 2	$2\frac{2}{4}$	2,5

FIGURE 25

A decimal fraction equivalent to a "common" fraction is obtained by dividing the numerator by the denominator. The children quickly learn to compare decimal fractions because they are part of the same number system as natural numbers and because they are often used in measurement. Changing fractions into decimal notation is thus another way of helping children to understand how to compare fractions. They could even use a calculator to perform the transformation. The advantage of calculators is that they expose children to less common fractions or fractions that would be unmanageable otherwise.

Arrange the following fractions in order from smallest to biggest:

$$\frac{2}{3} \cdot \frac{5}{8} \cdot \frac{11}{18} \cdot \frac{7}{11} \cdot \frac{7}{12} \cdot \frac{3}{5} \cdot \frac{4}{7}$$

$\frac{2}{3} = 0,\overline{6}$ $\qquad$ $\frac{7}{12} = 0,58\overline{3}$

$\frac{5}{8} = 0,625$ $\qquad$ $\frac{3}{5} = 0,6$

$\frac{11}{18} = 0,6\overline{1}$ $\qquad$ $\frac{4}{7} = 0,571\ldots$

$\frac{7}{11} = 0,\overline{63}$

thus

$$\frac{4}{7} < \frac{7}{12} < \frac{3}{5} < \frac{11}{18} < \frac{5}{8} < \frac{7}{11} < \frac{2}{3}$$

FIGURE 26

Using a calculator in this way definitely enriches the children's education by making it easier for them to tackle other types of problems. For example, they can look for regular patterns in the decimal representation of fractions, particularly in special cases such as thirds, sevenths, ninths, and thirteenths. They can also solve problems with multiple solutions such as that of finding a number of fractions situated between two given fractions. For example, find five fractions between 2/5 and 1/3? Since 2/5 = 0,4 and 1/3 = 0,3, we can easily find the following solutions:

$0,34 = \frac{34}{100} = \frac{17}{50}$

$0,35 = \frac{35}{100} = \frac{7}{20}$

$0,36 = \frac{36}{100} = \frac{9}{25}$

$0,37 = \frac{37}{100}$

$0,38 = \frac{19}{50}$

$0,344 = \ldots$

4

DECIMAL NUMBERS

4.1 Importance of Decimal Numbers

It has already been shown that rational numbers can be written as ordered pairs, in decimal notation, or as percents. According to the situation at hand, one symbol or the other will be used, each one having its own fields of application and its own advantages.

The program states that decimal numbers should be stressed because Québec has now adopted the International System of Units (SI). Although decimal numbers used to arise specifically in activities related to business or in dividing up gallons of gasoline, today they come up in all sectors of life.

Each of the booklets that make up the curriculum guide covers one of the themes in the program. In this booklet, rational numbers written in decimal form are the focus of a specific chapter. In practice, however, teaching is not broken down in this manner and the learning of decimal numbers could be more spread over the elementary school years, so that they are no longer covered in isolation or on an occasional basis.

In spite of these considerations, it is unlikely that decimal numbers will ever eliminate a/b-type fractions. There are situations in which the use of computational algorithms is more practical and simple with fractions in the form of ordered pairs than with fractions in decimal notation. (e.g. the operation of division, the expression of a probability).

Except for specific activities related to the search for regular patterns, pupils should not be expected to learn more than the thousandths place in relation to decimal numbers. For repeating decimals, the first three decimal places are usually regarded as being sufficiently precise.

4.2 Decimal Fractions

Decimal fractions form a specific class of fractions whose denominators are made up of tenths, hundredths or thousandths. The expression "decimal fraction" designates a form of writing which differs from decimal notation (e.g. 0,4) where the numbers have a place value. The learning of decimal numbers should be based on a good understanding of decimal fractions ($4/10 = 0,4$).

It is therefore important to provide a large number of activities that will lead the children to a better understanding of this subset of fractions. These activities could even be incorporated into the activities planned for the learning of fractions. For example, certain simple fractions could be transformed into decimal fractions since this constitutes a particular instance of looking for equivalent fractions.

4.3 Decimal Notation

Pupils usually have no difficulty expressing a fraction in its decimal notation because this conversion is based on the number system they learned when studying natural numbers. To make this transition

easier two types of approaches can be used. The approaches both differ from, and complement, each other. The approach adopted could either call on the children's deductive ability or lead them to identify, through an inductive process, the concepts underlying the decimal notation of fractions.

In the first case, the principles of the numeration system of natural numbers should be reviewed and the writing of decimal numbers presented as an extension of these principles.

$$8 \times 10 = 80$$
$$8 \times 1 = 8$$
$$8 \times \frac{1}{10} = 0,8$$

The role of the comma should be presented as that of a bridge, dividing line or limit between two parts of a number, that is, between its whole-number part and its fractional part. Thus 0,8 represents and is read as zero and eight tenths. The breaking up of a number into powers of ten, which was used for natural numbers, is of great help in this approach because it clearly illustrates the concept of expanded notation in the decimal number system.

$$123 = (1 \times 100) + (2 \times 10) + (3 \times 1)$$
$$123,45 = (1 \times 100) + (2 \times 10) + (3 \times 1) + (4 \times 1/10) + (5 \times 1/100)$$

In the second example, which falls into the inductive category, the children should be presented with as wide as possible a range of situations involving decimal numbers. By observing and analyzing these examples, they are led through inductive reasoning to discover the principles behind, and ways of writing, decimal numbers.

- a certain object costs $ 1.25;
- the baby weighed 3,970 kg at birth;
- if a dollar is divided among four persons, each person will receive twenty-five cents ($ 0.25);
- the average height of the pupils in a given class is 1,35 m;
- a certain runner won the race in a time of 10,535 s.

4.4 Division of Two Natural Numbers

The main focus thus far has been fractions that can be converted into terminating decimals. For fractions which correspond to repeating decimals the first three decimal places are sufficient to give a relatively precise approximation. Thus, we can write 1/3 = 0,333 if we agree that 333/999 is approximately equal to 333/1000.

$$\frac{1}{3} = \frac{333}{999}$$

$$\frac{333}{999} \approx \frac{333}{1000}$$

$$\frac{333}{1000} \rightarrow 0,333$$

We have already seen that a fraction in a/b form can express the division of the numerator by the denominator. Children can understand this concept easily when there is no remainder, as for example in 4/2 = (4 ÷2)=2. On the other hand, they will be less sure when presented with an example in which there is a remainder; in this case, it would be beneficial to help them express the remainder of a division in decimal form.

3,000 | 8	1,000 | 6
24 0,375	6 0,166
60	40
56	36
40	40
40	36
0	4

To 3 decimal places, the last remainder is ignored.

On several occasions, eighths have been used as examples. A number of simple fractions such as these have interesting decimal equivalents which the children would be well-advised to memorize.

$\frac{1}{8} = 0,125 = 12,5\ \%$

$\frac{2}{8} = \frac{1}{4} = 0,25 = 25\ \%$

$\frac{1}{3} = 0,333 = 33\ \frac{1}{3}\ \%$

$\frac{3}{8} = 0,375 = 37,5\ \%$

$\frac{4}{8} = \frac{2}{4} = \frac{1}{2} = 0,50 = 50\ \%$

5

OPERATIONS ON FRACTIONS

5.1 Concept of Operation

When learning about natural numbers, the children will have been led, through a great number of experiences involving manipulation and exploration, to understand the meaning of a mathematical operation. Without repeating the content of Booklet C on natural numbers, it seems important to specify here that during their initiation to computation the children should have established a relationship between concrete operations and mathematical operations. Thus, the concrete experiences which will have helped them acquire an understanding of the concept of operation in relation to the set of natural numbers will make it considerably easier for them to learn about operations with another reference set, in this case, rational numbers. They will be able to concentrate on doing operations involving fractions. They will have to discover, understand and assimilate the rules allowing them to perform operations. In other words, they will have to learn, after checking all the steps, to proceed in such a way as to ensure accurate results.[1]

In Chapter Two, emphasis was placed on the importance of presenting fractions as numbers that can be transformed by an operation. In this chapter, the difficulties particular to each operation will be examined, given that the focus is on "how to solve" a problem of the type $\frac{1}{2} + 1/3$.

5.2 Addition and Subtraction

Addition and subtraction of fractions are two operations which children understand without difficulty, probably through analogy to natural numbers. Two numbers representing two quantities (in this case at least one of the numbers will be a fraction) can be added to each other to form a third number. Subtraction involves the reverse of this operation.

Clearly the best teaching/learning situation is that which allows the children to discover by themselves the concepts and algorithms involved in an operation. What methods would best foster the learning of addition and subtraction?

It is advisable to start with the addition of natural numbers and to add in unknown elements progressively. Adding a natural number and a fraction usually presents no difficulty; nor does the addition of two fractions with a like denominator. With regard to these simple cases, the children will already have discovered the rules on their own before trying to perform these operations in school.

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1. Gaston Mialaret, L'apprentissage des mathématiques. (Brussels: Dessart, 1967) p. 65.

   This citation is a free translation.

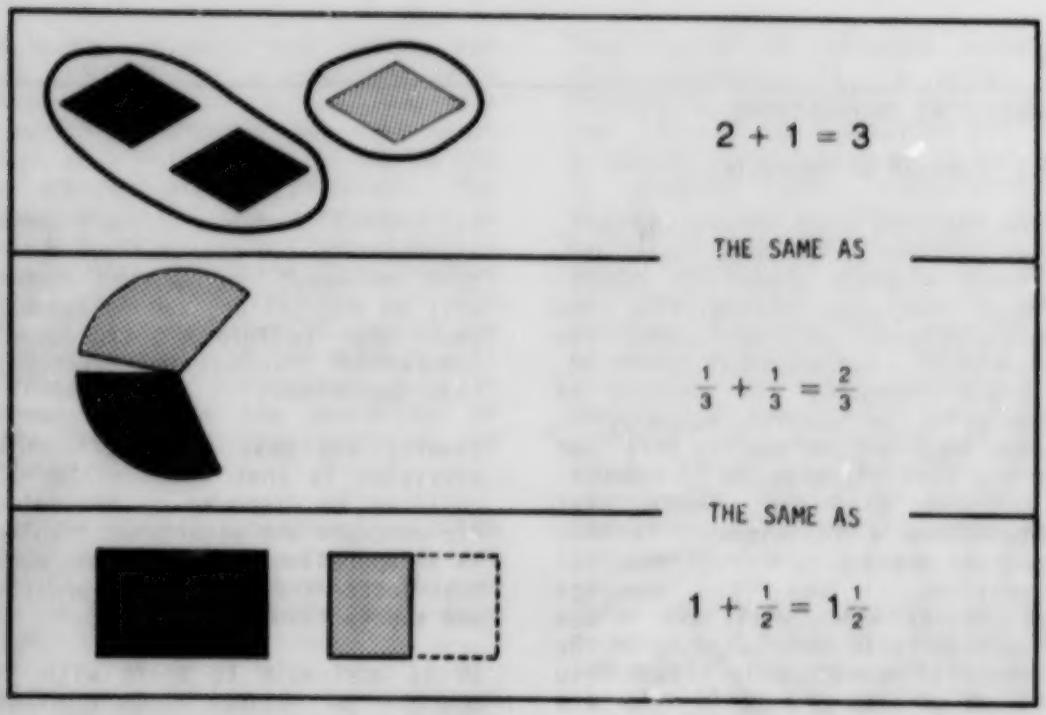


FIGURE 27

Children can be helped to perform operations on fractions using ordinary materials and illustrations. Even if they do not find the exact answers, they will see that they can eventually obtain a result which is larger than one and that there is a solution for every problem.

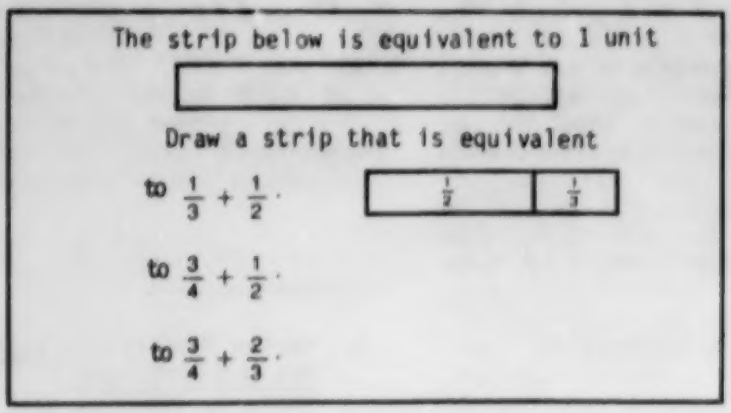

The strip below is equivalent to 1 unit
Draw a strip that is equivalent
to $\frac{1}{3} + \frac{1}{2}$.
$\frac{1}{2}$
$\frac{1}{3}$
to $\frac{3}{4} + \frac{1}{2}$.
to $\frac{3}{4} + \frac{2}{3}$.

FIGURE 28

When the children are looking for the elements of a solution to a problem such as 1/8 + 3/4, the teacher can provide them with means that will enable them to verify and to generalize their findings. The fraction table, for example, is an appropriate tool for helping children to use equivalent fractions. A review of the activities involving multiples and common multiples of two numbers will emphasize the predominant role of denominators in the operations of addition and subtraction on fractions.

Taking into consideration the reservations already expressed (pp. 15-17), the number line can be used to make these operations more concrete in certain cases.

When the children have discovered the principles and general rules for adding and subtracting fractions, the teacher could review with them the different cases that can be solved in more than one way.

For example, the following cases could be reviewed:

- 2 whole numbers expressed in the form of fractions;
- 1 whole number and one fraction;
- 2 fractions with like denominators;
- 2 fractions with unlike denominators, one of which is a multiple of the other;
- 2 fractions whose denominators have no common factor except 1;

* $\frac{1}{2} + \frac{1}{3}$;

* $\frac{1}{2} + 1\frac{1}{3}$;

* $1\frac{1}{3} + 2\frac{1}{4}$.

This review of different examples (not necessarily in order of increasing difficulty) will allow the children to synthesize all the processes they have used so far and to develop their computational skills. Adding fractions with unlike denominators is certainly the major stumbling block in this learning process. Solving such problems requires the coordination of many operations: finding the multiples of each denominator; determining a common multiple, preferably the lowest one; changing each fraction to the same denominator; adding the numerators; and lastly, if necessary, simplifying the results.

This complexity points out the necessity of using the simplest and commonest fractions. On the other hand, a child may prefer to use a common denominator that is not the lowest one. In some respects, this method may seem simple but it often implies a larger denominator and hence a result that can be simplified. Even if this simplification process offers certain advantages, it is unnecessary since the equation 1/4 + 1/6 = 10/24 is as accurate as 1/4 + 1/6 = 5/12.

Adding three fractions with unlike denominators is an excellent exercise for applying the methods of calculation and concepts acquired. It is also a good occasion for presenting the commutative and associative properties of addition.

Subtracting fractional numbers (3 1/3 - 5/8) sometimes presents difficulties as well and deserves some attention. This opportunity could be used to transform the terms in question into fractional expressions and, if necessary, the fractional expression of the result into a fractional number, as shown in the following:

$$8\frac{1}{4} - 5\frac{3}{4} = \frac{33}{4} - \frac{23}{4}$$
$$= \frac{10}{4}$$
$$= 2\frac{1}{2}$$

The aim is not, however, to systematically eliminate subtractions that involve borrowing. Both processes constitute complementary approaches that enrich the child's education.

5.3 Multiplication

Once again, an analogy should be drawn between natural numbers and rational numbers with regard to multiplication. The same mathematical principle is involved in that two numbers are related by an operation and produce another number. Also, when represented graphically or applied to a concrete situation, the two factors can be seen to play different roles, depending on whether they are multiplier or multiplicand. In fact, the same duality always exists between the physical model and the mathematical model of an operation. This apparent opposition nevertheless makes it possible to develop certain procedures apt to reinforce the child's understanding of the concept of fraction.

Language can be used to resolve the dilemma and facilitate the learning of this operation. For example, it is easier to understand the meaning of:

- three-quarters of 12 than (3/4 x 12);
- what is half of 1/4? than (1/2 x 1/4).

It is therefore important to help the children describe in words the operations they are being asked to perform.

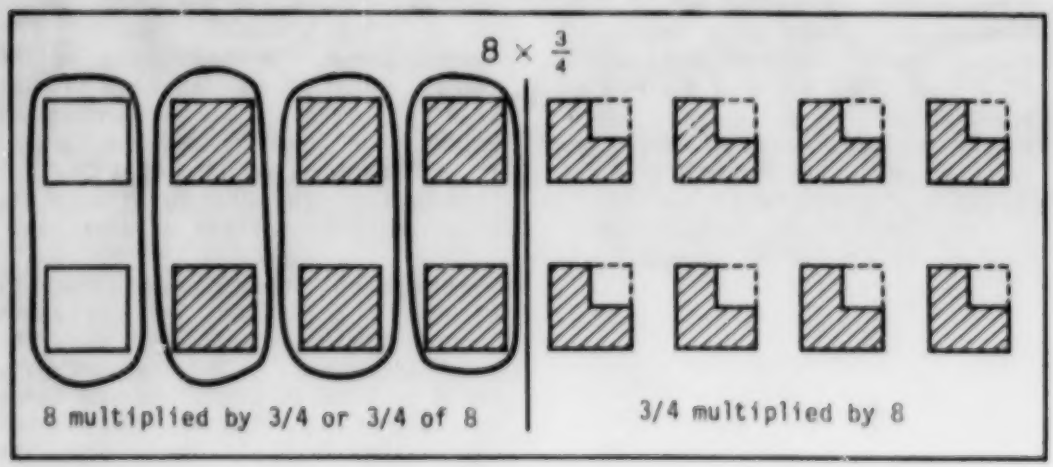


FIGURE 29

The table of fractions previously referred to is another means of increasing comprehension and of providing opportunities for practicing multiplication. It must, however, be used with a great deal of care. It is an activity similar to that which consists of successively folding a sheet of paper that has been chosen to represent the unit.

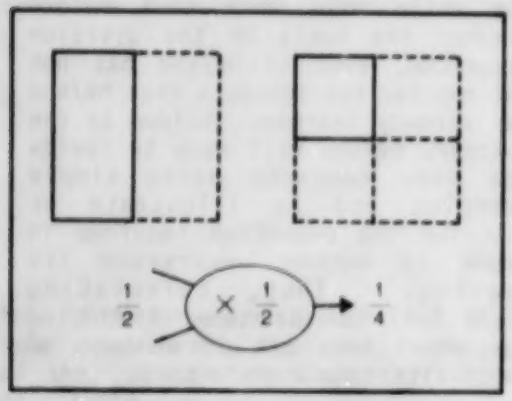


FIGURE 30

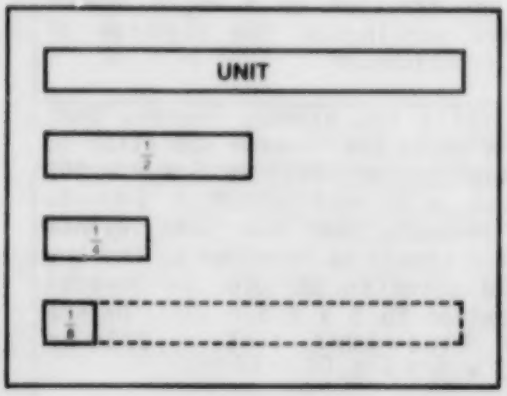


FIGURE 31

Another meaning can also be given to the word fraction (see Section 2.4), and multiplication then presented as a succession of two operations: division followed by multiplication. Thus, the operation 8 x 3/4 can be presented as follows:

$$8 \times \frac{3}{4} = 8\ (\div 4)\ (\times 3) = 6$$

or
$8\ (\div 4) = 2$
and
$2\ (\times 3) = 6$

Finally, use can be made of "function machines," which show fractions as functions or operators capable of transforming quantities. Indirectly, the children review the methods of computation they have learned up to then and construct an algorithm that they can apply in a more general way to all cases of multiplication.

$$\frac{3}{4} \times \frac{8}{1};\ \left(\frac{3}{4} \times\right)\ \frac{1}{2};\ \frac{2}{1} \times \frac{1}{4}$$

5.4 Division as the Inverse Operation to Multiplication

The division of fractions is a special case in that presenting it as a repeated subtraction does not make it any easier to understand. Thus, any analogies with natural numbers must be drawn with care. Whatever instructional approach is adopted in the classroom, the division of fractions is an operation which is difficult to render concrete and the result of which is not always evident.

Since division is an inherently complex operation, it should be broken down into steps that the child can solve by himself/herself. In this sense, the child should be helped to observe the effect on the value of a fraction of dividing or multiplying one of its terms by a

whole number. He/she will thus be able to deduce that by multiplying the numerator or by dividing the denominator, the fraction can be made larger and that it can be made smaller by performing inverse operations. The child must thoroughly understand the division of one fraction by a whole number before working on the division of two fractions.

The child has already learned that division is the inverse operation to multiplication; just as $5 \times 2 = 10$, so $10 \div 5 = 2$ or $10 \div 2 = 5$. The concepts that have been learned so far should be referred to and the child asked to perform the inverse operation to $5 \times 2/3 = 10/3$; he/she may find $10/3 \div 5 = 2/3$ or $10/3 \div 2/3 = 5$.

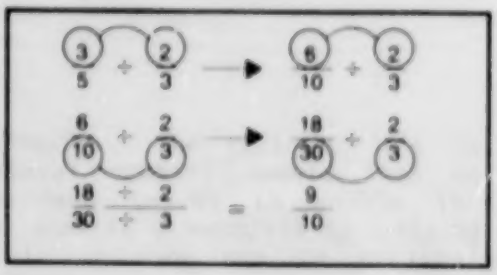
This discovery, although simple in appearance, contains all the elements that will enable pupils to explore and understand the computational processes involved in this complex operation. The challenge should not be underestimated: it is no easy matter for secondary school or college students, or indeed for adults, let alone for elementary school pupils, to understand all the abstract complexities of this operation.

The child can then be asked to go back to the beginning of his/her work, replacing 5 by its equivalent 5/1; he/she will then have:
$5/1 \times 2/3 = 10/3$
thus $10/3 \div 5/1 = 2/3$
and $10/3 \div 2/3 = 5/1 = 5$

The child will then have before him/her the basis of the division algorithm, even if he/she has not yet applied the concepts that he/she has already learned. Guided by the teacher, he/she will have to review the same concepts using simple examples and to illustrate or describe the operation required in order to better understand its meaning. Thus, calculating $(1 \div 1/4)$ is the same as finding how many times 1/4 of a sheet of paper fits into a whole sheet.

The child will eventually come up with a method of computation that will not necessarily be the same as that of his/her neighbour and that may not even be a general rule. It is therefore worthwhile for teacher and pupil to go over the different cases of division and the possible algorithms together. This chapter ends with an illustration of the three main algorithms that can be used to divide fractions.

Division by homologous terms: when necessary, it is possible to find a fraction equivalent to 3/5 that makes the 2 divisions possible.



Both fractions are brought to the same denominator, then the division of the two numerators is expressed as a ratio.

$$\frac{4}{5} \div \frac{2}{3} = \frac{12}{15} \div \frac{10}{15}$$

$$= \frac{12 \times 15}{15} \div \frac{10 \times 15}{15}$$

$$= 12 \div 10$$

$$= \frac{12}{10} \text{ OR } \frac{6}{5}$$

The first fraction is multiplied by the inverse of the second fraction.

$$\frac{4}{5} \div \frac{2}{3} = \frac{4}{5} \times \frac{3}{2}$$

$$= \frac{12}{10}$$

$$= \frac{6}{5}$$

OPERATIONS ON DECIMAL NUMBERS

6.1 Equality and Inequality of Decimal Numbers

When beginning the study of this concept, all that some children require is a review of the natural number system. Other children are confused by comparing two numbers which have different numbers of decimals. The role of "0"s in natural numbers should then be reviewed so as to enable the children to extrapolate by applying their observations to rational numbers.

Indicate the correct sign between the numbers shown. < = >		
410 ◯ 410,00	3,19 ◯ 3,2	
345,6 ◯ 34,56	1,22 ◯ 1,190	
23 ◯ 23,001	2,002 ◯ 2,1	
2,4 ◯ 2,40	0,2 ◯ 0,201	

FIGURE 32

To call the child's attention to place value, he/she can be asked to break numbers down into powers of ten or simply to bring all numbers to the same number of decimal places by adding "0" whenever necessary:

0,191 < 0,2 because 0,191 < 0,200

There is a distinct advantage to using simple examples as a point of departure and gradually increasing their difficulty until numbers with like notations can be compared. These activities focus attention on the structure of place value and thereby increase pupils' understanding of decimal numbers in general.

6.2 Addition and Subtraction

Once the children have understood that a comma is used to separate the whole part of a number from its fractional part, they discover through simple and concrete situations that the different units must be correctly aligned to perform an addition or a subtraction. The algorithm is nothing more than an extension of computation with natural numbers. These operations will therefore be covered very briefly.

Whenever possible, this learning must be related to the children's experiences by including units of

measure, of money, etc. Estimating results plays an important role in understanding these operations, because by trying to determine the order of size of the result, the child must distinguish the whole number part from the fractional part of the numbers in question. In the example 13,4 m + 3,56 m = _____, the child who estimates 48 m as an answer has overlooked the comma, because the answer will in fact be only little more than 16 (13 + 3).

While the addition and subtraction algorithms for fractions in a/b form are not easily mastered, and while the use of the addition algorithm for fractions in the form of ordered pairs must be limited to simple fractions, it quickly becomes apparent that the opposite is so for decimal numbers. Using decimal notation for fractions also has the advantage that it uses the same language as most calculators.

The teacher's aim should be to develop first the children's ability to estimate and then their computational skills. In order to do so, a variety of stimulating situations should be used to enable them to perform a wide range of operations on a regular basis. Whenever possible, these situations should be integrated into computation exercises involving natural numbers.

6.3 Multiplication

There are two different levels of difficulty in multiplication depending on whether the decimal number is multiplied by a whole number or by another decimal number. The first case can be considered as a repeated addition and does not present more difficulty than the multiplication of two natural numbers. Special attention should be paid to the multiplication of a decimal number by 10, 100, or 1 000. The development of this skill will make it easier for pupils to learn the division algorithm for decimal numbers by eliminating the fractional part of decimal numbers.

2 x $ 3.75 = $ 3.75 + $ 3.75 =
$ 7.50

10 x 2,24 m = 22,4 m
100 x 2,24 m = 224 m

It can be seen that these examples misuse symbols in that units of measure should never be used with mathematical operations in the same proposition. Instead, to find the total length of 10 segments measuring 2,24 m each, we should say: "I perform the operation 10 x 2,24 = 22,4." Such a use of symbols is nevertheless acceptable on occasion if it helps to eliminate confusion and to enhance the pupil's understanding.

In order to develop a good understanding of the multiplication of two decimal numbers, pupils should review the multiplication of fractions in a/b form, and in particular the multiplication of decimal fractions. They will already have learned to change fractions in a/10 form to decimal notation. Through observation, they will be able to discover, verify, generalize, and apply a rule that makes it possible to multiply two decimal numbers directly.

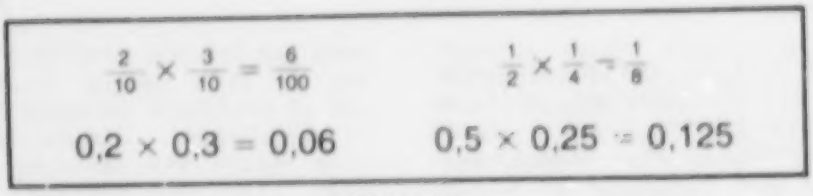


FIGURE 33

In most cases, and particularly in cases where the whole-number part of the factors is other than zero, children must be helped to estimate the size of their product. They can also be helped to transform decimal numbers into rational numbers by using a rule that they have already learned

$$2,5 \times 3,2 = \frac{25}{10} \times \frac{32}{10}$$

$$= \frac{800}{100}$$

$$= 8.$$

The following are particular examples involving the multiplication of decimal numbers. They make it possible for learners to verify or apply some of what they have learned or to choose the particular notation for a fraction which makes computation easier:

- multiply a number by 0,9
- multiply a number by 11; 1,1; 2,1...
- multiply a number by 0,5
- multiply a number by 0,25 or 0,75
- multiply a number by 0,125.

6.4 Division

Learning to divide decimal numbers seems difficult when approached in a technical way. It can be made easier by helping children to understand the computational processes involved.

The operation of dividing natural numbers should be reviewed and pupils made aware of what happens to the quotient when the dividend of the divisor is multiplied by 10. In other words, the value of a division equation is not changed if the dividend and the quotient are multiplied by the same number. On the other hand, if the divisor is multiplied by 10, the quotient must be divided by 10 to keep the same ratio. When the children have mastered these transformations, when they are able to find the correct answer after having performed a transformation that makes computation easier, they can be helped to solve more difficult operations. With the teacher's help, they will be in a position to discover computational algorithms that they will be able to verify by applying them to the following five types of situations:

- 1 ÷ 5
- 24,6 ÷ 6
- 246 ÷ 4,1
- 125,4 ÷ 4,18
- 12,54 ÷ 41,8

Finally, the skill of roughly estimating the whole-number part of the quotient usually makes it possible to place the comma in the right place in the quotient after having performed a division without any commas. In a more general way, estimation is a skill that is important to develop regardless of

the operation, since it makes it possible to detect errors in calculation:

- In 16,72 ÷ 4, the whole-number part of the quotient will be somewhere between 1 and 9 (0 and 10 excluded) since $1 < 16/4 < 10$.
- In 167,2 ÷ 4,18, the quotient will be 40 because $10 < 167/4 < 100$.
- In 17,25 ÷ 2,3, the whole-number part of the quotient will have only one numeral since $17/2 \approx 8$.
- In 225,804 ÷ 19,1, the whole-number part of the quotient will have 2 numerals because $225/19 \approx 200/20 = 11$.

CONCLUSION

The purpose of this booklet has been to present a global view of the process of teaching fractions in the elementary school. Hopefully it will have provided the reader with food for thought and will help teachers to reassess their teaching in light of all the knowledge and skills that the child is expected to acquire in the process of learning fractions.

Changes have taken place in both the teaching of fractions and elementary school teaching practices. It is hoped that this booklet sufficiently clarifies this part of the mathematics program so that in the planning of pedagogical activities, the major orientations which have been elaborated will be respected and the specific needs of each milieu taken into consideration.

One of the basic concerns is that the teaching of fractions not take place too early in order to ensure the appropriateness of the instructional objectives for the child's level of intellectual development. Another is that a balance be established among the activities aimed at the assimilation of content, those aimed at the acquisition of algorithms, and those which are merely exploratory.

It is impossible to overemphasize the importance, indeed the necessity, of helping children develop a clear understanding of the concept of fraction before they tackle more complex notions. This understanding can only be developed after the children have grasped the fundamental difference between the basic meaning of a fraction (a number) and all the ways in which it is applied.

Mention should also be made of the necessity of developing the children's computational skills in operations involving fractions. The development of these automatisms cannot be separated from the understanding, assimilation, and memorization of the fundamental algorithms used in computation.

Finally, we wish to point out again the increasing emphasis that should be placed on decimal numbers in the teaching of mathematics, with the proviso that this learning be based on a sound knowledge of simple fractions.